Solutions and Comments on Homework 3
Decimals and Irrational numbers

- **Looking at your charts for the period of the fraction $\frac{1}{n}$, what can you say about the period compared to n?**

  In general, the period is always less than the denominator. You might also notice that in the case where the denominator b is prime, the period divides $b-1$, although this is harder. It is a consequence of Fermat's Little Theorem, which states that if p is a prime and $p \nmid a$, then $a^{p-1} \equiv 1 \bmod p$. Thus, if $p \nmid 10$, it follows that $10^{p-1} \equiv 1 \bmod p$. Hence, the remainder in the $p-1$st step of our algorithm for finding decimals has become equal to 1, which for $\frac{1}{p}$ is the starting remainder, and as we shall see below, this forces the decimal to repeat. In general, one can use Euler's generalization to Fermat's Little Theorem. This states that if $\gcd(a,b) = 1$ then $a^{\phi(b)} \equiv 1 \bmod b$, where $\phi(b)$ is the number of positive integers less than b that are relatively prime to b. Again, if b is not divisible by 2 or 5, we have that the period of $1/b$ must divide $\phi(b)$.

  It is not true, however, that the period must equal either $p-1$ in the prime case or $\phi(b)$ in the other case as we can see by $1/37$ is 3 and not 36.

- **In the algorithm for finding the decimal expansion of a fraction, how many different remainders can appear? What happens once the same remainder shows up twice?**

  Since we are dividing by the denominator, b, at each stage, and the division algorithm tells us that the possible remainders are non-negative integers that are less than b, the possibilities for the remainders are $0, 1, \ldots, b-1$. Consequently there are b possible remainders.

  Once the remainder repeats, the decimal repeats as we shall prove below.

- **Prove that the period of the fraction $\frac{a}{b}$ (a and b positive integers) must be less than b.**

In this case, we must assume $b > 1$ (or say that a terminating decimal has period 0 rather than 1). Let $q_0, q_1, \ldots$ be the quotients in our algorithm (see the text for the algorithm) and let $r_0 = a$, and $r_1, r_2, \ldots$ be the remainders, so that

$$\begin{aligned} a &= q_0 b + r_1 \\ 10 r_1 &= q_1 b + r_2 \\ 10 r_2 &= q_2 b + r_3, \end{aligned}$$

and so on. First note that if any of the remainders are ever equal to 0, all subsequent remainders must be equal to 0, so that the period is 1 and hence is less than b.

Assuming that $r_i \neq 0$ for all i, we have that two of $r_0, r_1, \ldots, r_{b-1}$ must be equal as there are only $b - 1$ non-zero remainders. Call these r_k and r_l and suppose $l < k$, so that $t = k - l > 0$. We claim that for all $j \geq l$ $r_j = r_{j+t}$. To see this, note that the result is true for $j = l$ by our choice of l and k. Assuming it is true for some fixed $j > l$, by definition of our algorithm we have that

$$\begin{aligned} 10 r_j &= q_{j+1} b + r_{j+1} \qquad \text{and} \\ 10 r_j = r_{j+t} &= q_{j+t+1} b + r_{j+t+1}, \end{aligned}$$

where $0 \leq r_j, r_{j+t+1} < b$. Thus (q_{j+1}, r_{j+1}) and (q_{j+t+1}, r_{j+t+1}) are both solutions to the divison algorithm for $10 r_j$ divided by b. However the division algorithm says any two such solutions are equal so that $r_{j+1} = r_{j+t+1}$. The principle of mathematical induction now implies that $r_j = r_{j+t}$ for all $j \geq l$. Using the uniqueness part of the division algorithm result again, however, we see that $q_{j+1} = q_{j+t+1}$ for all $j \geq l$. Thus the period of a/b is less than or equal to t. (We need that t is the smallest such repetition to assure that the period equals t.)

- **Show that you can multiply two irrational numbers and get a rational product out. What does this say about the irrational numbers and the closure laws?**

By definition, $\sqrt{2}\sqrt{2} = 2$. Hence the irrational numbers are not closed under multiplication, since $\sqrt{2}$ is irrational. As $-\sqrt{2}$ is irrational, we also have that the irrational numbers are not closed under addition either since $\sqrt{2} + (-\sqrt{2}) = 0$ is rational.

- We would like to show that one can raise an irrational number to an irrational power and get a rational number out. Consider $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$. Suppose the first of these is irrational. Show that the second gives an example of an irrational number raised to an irrational power yielding a rational number.

  Calculating, we have $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$. Thus it is rational. If we suppose that $\sqrt{2}^{\sqrt{2}}$ is irrational, then we would have an example of an irrational number raised to an irrational power giving a rational number.

- Using the above, argue that there exist irrational numbers a and b such that a^b is rational.

  We will prove that there exists a pair (a, b) of irrational numbers such that a^b is rational. We will not construct the pair, however. First, suppose $\sqrt{2}^{\sqrt{2}}$ is rational. Then we have an example taking $a = b = \sqrt{2}$ since we know by class that $\sqrt{2}$ is irrational.

  On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, the previous problem gives us that choosing $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ then a^b is rational with each irrational.
  Q.E.D.

  The question is $2^{\sqrt{2}}$ rational, algebraic, or transcendental? is a very important question in 20th century mathematics. David Hilbert announced it as the 10th of his 23 problems at the International Mathematics Congress of 1900. It wasn't solved for about 30 years until Gelfand and Schneider proved that $2^{\sqrt{2}}$ is transcendental. They showed something considerably harder, which we will discuss later in the term.

- Suppose you have four fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{x}{y}$ and $\frac{z}{w}$. Suppose further that $\frac{a}{b} > \frac{x}{y}$ and $\frac{c}{d} > \frac{z}{w}$. What can you say about $\frac{a+c}{b+d}$ and $\frac{x+z}{y+w}$.

  Basically, we can't say anything. There exist examples where $\frac{a+c}{b+d} > \frac{x+z}{y+w}$ and vice versa. See below for the less obvious one. In fact, you can find examples where you get equality.

- **In 1983, Ken Oberkfell went** 143 **for** 488 **for the season, and in 1984, Ken Oberkfell went** 87 **for** 324 **for the season. Mike Scioscia was** 11 **for** 35 **in 1983 and** 93 **for** 341 **in 1984. Compare their batting averages for 1983, 1984, and for the two years together. Does this surprise you? This is an example of Simpson's paradox on fractions.**

We have the following chart

Player	1983	1984	combined
Oberkfell	.293	.269	.283
Scioscia	.314	.273	.277

This is a surprise to most people because they usually misunderstand averaging fractions and the importance of the numerator and the denominator in applications. At some level, this is really an example of why we don't want to work solely with decimals. If I had just written down the chart above, almost everyone would have been surprised by the result. Decimals often obscure the dependence of a fraction on the numerator and denominator. As most of us saw in the earlier problem on married couples, this confusion can seriously complicate our student's mathematical reasoning.